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On the ν-Dimensional One-Component Classical Plasma: The Thermodynamic Limit Problem Revisited

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We prove the H-stability property and the existence of the thermodynamic limit of the free energy density of the two-dimensional, one-component classical plasma. We give lower and upper bounds on the free energy density in any dimension ν and draw some consequences.

KEY WORDS: H-stability; thermodynamic limit; superharmonicity; "cheese" theorem; scaling properties; two-component plasma; statistical mechanics.

1. INTRODUCTION

This paper deals with the one-component classical plasma constituted, in a domain Λ, by N point charges (electrical charge equal to $-e$), immersed in a uniform neutralizing background of density $\rho = N/|\Lambda|$.

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We prove the H-stability property of the Hamiltonian as well as the existence of the thermodynamic limit of the canonical free energy density in the two-dimensional case. Accurate extensive bounds on the free energy are established in any dimension ν, and some bounds on the pressure are obtained. The peculiarity of the two-dimensional case, due to the long-range nature of the logarithmic Coulomb potential, is discussed. The three-dimensional case has been published recently by Lieb and Narnhofer.[1]

2. H-STABILITY

Theorem 1. Assuming the system to be neutral, with ρ the fixed background density, then, for any dimension ν, there exist extensive *density-dependent* lower bounds on the Hamiltonian:

$$H^{(\nu)}(x_1,\ldots,x_N) \geqslant -N\mathbf{b}^{(\nu)}(\rho), \qquad \mathbf{b}^{(\nu)}(\rho) = \tfrac{1}{2}\nu E^{(\nu)}(\rho) + \tfrac{1}{4}\delta_{\nu,2}$$

where $E^{(\nu)}(\rho)$ is the self-energy of a ball of density ρ and total charge $+e$, i.e. $[S(\nu) \equiv 2\pi^{\nu/2}\Gamma(\nu/2)]$,

$$E^{(\nu)}(\rho) = \begin{cases} \dfrac{2}{\nu}\left[|\nu-2|\dfrac{\nu+1}{\nu+2} + \operatorname{sign}(\nu-2)\right]\dfrac{e^2}{2}\left(\dfrac{\rho S(\nu)}{\nu}\right)^{1-(2/\nu)}, & \nu \neq 2 \\ \dfrac{2}{\nu}\left(\dfrac{\nu-1}{\nu+2} + \dfrac{1}{2}\ln\dfrac{\rho S(\nu)}{v}\right)\dfrac{e^2}{2}, & \nu = 2 \end{cases}$$

We have previously given[2] a heuristic derivation of these bounds, as well as a careful analysis of the nature of the ground state of the model in two and three dimensions; $-\mathbf{b}^{(\nu)}(\rho)$ is simply the Coulomb energy of the neutral elementary system (consisting of just one particle and the background) in the configuration of smallest energy. Explicitly,

$$\mathbf{b}^{(1)}(\rho) = -\frac{e^2}{12\rho}, \qquad \mathbf{b}^{(2)}(\rho) = e^2\left(\frac{3}{8} + \frac{1}{4}\ln \pi\rho\right), \qquad \mathbf{b}^{(3)}(\rho) = \frac{9e^2}{10}\left(\frac{4\pi\rho}{3}\right)^{1/3}$$

This theorem has been proved for the three-dimensional case by Lieb and Narnhofer;[1] we give here the proof for the two-dimensional case along the same lines.

Proof. Following an idea of Onsager,[3] we replace the point charges by charge distributions smeared into disks of radius a. We define $H_{bb}^{(2)}$, the self-energy of the background in the domain; $H_i^{(2)}$, the interaction energy of the particle i with the background; $H_{ij}^{(2)}$, the interaction energy of the particles i and j; $\hat{H}_{ij}^{(2)}$, the interaction energy (or twice the self-energy when $i = j$) of disks of total charge $-e$, centered at x_i and x_j; and $\hat{H}_i^{(2)}$, the interaction energy of such a disk with the background. Then

$$H^{(2)}(x_1,\dots,x_N) = H_{bb}^{(2)} + \sum_{i=j}^{N} H_i^{(2)} + \sum_{i<j}^{N} H_{ij}^{(2)}$$

$$= \left\{ H_{bb}^{(2)} + \sum_{i=1}^{N} \hat{H}_i^{(2)} + \frac{1}{2} \sum_{i,j=1}^{N} \hat{H}_{ij}^{(2)} \right\}$$

$$+ \frac{1}{2} \sum_{i\neq j}^{N} (H_{ij}^{(2)} - \hat{H}_{ij}^{(2)})$$

$$+ \sum_{i=1}^{N} (H_i^{(2)} - \hat{H}_i^{(2)}) - \frac{1}{2} \sum_{i=1}^{N} \hat{H}_{ii}^{(2)}$$

The quantity in the brackets is obviously positive. On the other hand, $H_{ij}^{(2)}$ is greater than or equal to $\hat{H}_{ij}^{(2)}$ by virtue of the superharmonicity of the Coulomb potential, so that

$$H^{(2)}(x_1,\dots,x_N) \geqslant \sum_{i=1}^{N} (H_i^{(2)} - \hat{H}_i^{(2)}) - \frac{1}{2} \sum_{i=1}^{N} \hat{H}_{ii}^{(2)}$$

A direct calculation shows that $\hat{H}_{ii}^{(2)} = e^2(\frac{1}{4} - \ln a)$, and $H_i^{(2)} - \hat{H}_i^{(2)} \geqslant -\pi\rho a^2 e^2/4$. The best bound on $H^{(2)}(x_1,\dots,x_N)$ is obtained when $\pi\rho a^2 = 1$. Consequently,

$$H^{(2)}(x_1,\dots,x_N) \geqslant -Ne^2(\tfrac{3}{8} + \tfrac{1}{4}\ln \pi\rho)$$

for any configuration $(x_1,\dots,x_N)$ of particles.

Remark. These bounds hold for each domain Λ of reasonable shape. However, except for the case $N = 1$ particle and Λ spherical, where the ground state has the greatest symmetry, they can never be reached. For $N > 1$ and large, the configurations of highest symmetry are then given by the Wigner lattices,[2] i.e., the crystalline configurations of particles whose energies are extremely close to the above bounds. Finally, these bounds provide *lower* bounds on the free energy density, i.e.,

$$\beta f^{(\nu)} \geqslant -\rho[1 + \beta\rho \mathbf{b}^{(\nu)}(\rho) - \ln \rho] \tag{1}$$

3. UPPER BOUNDS ON THE FREE ENERGY DENSITY

Theorem 2. Assuming always charge neutrality, there exists an *extensive* upper bound on the canonical free energy, i.e.,

$$\beta f^{(\nu)} \leqslant \rho \ln \rho - \beta\rho\alpha(\rho) \tag{2}$$

To prove this, we use the method of cells worked out in our previous paper.

Proof. Let Λ be a domain of reasonable shape. Then, it is possible to partition Λ into N cells $\mathbf{c}_i$, $(i = 1,\dots, N)$, with $\mathbf{c}_i \cap \mathbf{c}_j = \varnothing$, $i \neq j$, and

$\bigcup_i \mathbf{c}_i = \Lambda$, each one of volume $1/\rho$ and of *finite* self-energy. Each cell contains just one particle and thus is electrically neutral. Restricting ourselves to such configurations, we have, using Jensen's inequality, $[(x) \equiv (x_1,\ldots, x_N)]$,

$$\int d^\nu x_1 \cdots d^\nu x_N \exp[-\beta H^{(\nu)}(x)] \geqslant N! \int_{(\text{cells})} d^\nu x_1 \cdots d^\nu x_N \exp[-\beta H^{(\nu)}(x)]$$
$$\geqslant N! \left(\frac{1}{\rho}\right)^N \exp\{-\beta \langle H^{(\nu)}\rangle_{\text{cell,free}}\}$$

On the other hand, for $\varphi^{(\nu)}(|x|)$ the Coulomb potential in ν dimensions,

$$\langle H^{(\nu)}\rangle_{\text{cell,free}} = H_{bb}^{(\nu)} + e^2 \sum_{i<j}^{N} \frac{\iint_{(\text{cells})} d^\nu x_i\, d^\nu x_j\, \varphi^{(\nu)}(|x_i - x_j|)}{\iint_{(\text{cells})} d^\nu x_i\, d^\nu x_j}$$
$$- e^2\rho \sum_{i,j=1}^{N} \frac{\iint_{(\text{cells})} d^\nu x_i\, d^\nu x_j \varphi^{(\nu)}(|x_i - x_j|)}{\int_{(\text{cells})} d^\nu x_i}$$
$$= -\sum_{K=1}^{N} \text{Self}_K^{(\nu)}$$

where

$$\text{Self}_K^{(\nu)} \equiv \tfrac{1}{2}\rho^2 e^2 \iint_{(\text{cell } K)} d^\nu x\, d^\nu y\, \varphi^{(\nu)}(|x - y|)$$

is the self-energy of the cell K. In particular, for a large domain, a choice of cells of Wigner–Seitz type is possible, and we have

$$\beta f^{(\nu)} \leqslant \rho \ln \rho - \beta\rho\, \text{Self}_{WS}^{(\nu)}$$

The theorem is proved by putting $\alpha(\rho) = \text{Self}_{WS}^{(\nu)}$.

Consequently, combining the inequalities (1) and (2), we obtain for the excess free energy density $\Delta f^{(\nu)} \equiv f^{(\nu)} - f_0$, where $f_0 = \beta^{-1}\rho(\ln \rho - 1)$,

$$-\beta \mathbf{b}^{(\nu)}(\rho) \leqslant (\beta\, \Delta f^{(\nu)})/\rho \leqslant 1 - \beta\, \text{Self}_{WS}^{(\nu)} \tag{3}$$

The better upper bounds are obtained by taking the Wigner–Seitz cells of maximum self-energy (hence associated with the configuration of particles with the minimum energy), i.e., for $\nu = 2$, the hexagonal cell, and for $\nu = 3$, that of the bcc lattice. From the results derived in Ref. 2, we obtain explicitly, for $\nu = 1, 2, 3$,

$$\beta e^2/12\rho \leqslant (\beta\, \Delta f^{(1)})/\rho \leqslant 1 + (\beta e^2/6\rho)$$
$$-\beta e^2(\tfrac{3}{8} + \tfrac{1}{4}\ln \pi\rho) \leqslant (\beta\, \Delta f^{(2)})/\rho \leqslant -\beta e^2(\tfrac{3}{8} + \tfrac{1}{4}\ln \pi\rho) + 1.253$$
$$-0.9\beta e^2(4\pi\rho/3)^{1/3} \leqslant (\beta\, \Delta f^{(3)})/\rho \leqslant -0.66\beta e^2(4\pi\rho/3)^{1/3} + 1$$

Remarks. (1) The existence of these bounds, although not sufficient, clearly indicates and strongly suggests that the usual thermodynamic limit of the free energy density of this model may exist. (2) From these inequalities, there clearly emerges the peculiarity of the two-dimensional case: The lower and upper bounds have the same dependence with respect to the density ρ. This fact will be reflected in the equation of state (Section 5).

4. THE THERMODYNAMIC LIMIT

We shall give the proof of the existence of the thermodynamic limit for the free energy density for $\nu = 2$ dimensions. The idea is to employ the "cheese" theorem of Lieb and Lebowitz[4] adapted to the one-component case. The two-dimensional case presents a particular difficulty due to the long-range nature of the logarithmic potential. We shall formulate the problem for circular domains only. The case of more general domains can be handled along the customary lines given by Fischer[5] and Lieb and Lebowitz.

Consider a standard sequence of disks $\{\Lambda_k\}_{k=0}^{\infty}$, *strictly neutral*, of radii $R_k = R_0(1 + p)^k$, $p = 10$, where $R_0 = (\pi\rho)^{-1/2}$ is the radius of the disk Λ_0, which contains one particle in the neutralizing background of fixed density ρ. The number of particles in the disk Λ_k is then taken to be equal to $(1 + p)^{2k}$. Let $n_k \equiv \gamma^k(1 + p)^{2k}/p$, where $\gamma \equiv p/(1 + p)$.

According to Ref. 4, it is possible to pack Λ_l ($l > 0$) with $\bigcup_{j=0}^{l-1}$ (n_{l-j} disks Λ_j that do not overlap). Let us call D_l the complement of the union of the Λ_j ($j = 0,..., l - 1$) in Λ_l. The fraction of volume $A_l \equiv |D_l|/|\Lambda_l|$ left unfilled after the disk of type l has been packed,

$$A_l = 1 - \sum_{j=0}^{l-1} n_{l-j}(|\Lambda_j|/|\Lambda_l|) = \gamma^l, \qquad \gamma < 1$$

goes exponentially to zero; and the packing is *complete*.

Let $Z_l^{(2)}(\Lambda_l, N_l = \rho|\Lambda_l|; \beta)$, $l = 0, 1,...$ be the configuration partition function of the system in Λ_l with $N_l = \rho|\Lambda_l|$ particles in the background,

$$Z_l^{(2)}(\Lambda_l, N_l; \beta) = \frac{1}{N_l!}\int\cdots\int_{(\Lambda_l)^{N_l}} d^2x_1 \cdots d^2x_{N_l} \exp[-\beta H^{(2)}(x_1,..., x_{N_l})]$$

and the corresponding free energy density is $f_l^{(2)}$:

$$g_l^{(2)} = -\beta f_l^{(2)} = (\ln Z_l^{(2)})/|\Lambda_l|$$

Exploiting Newton's electrostatic theorem and the fact that all the domains, except D_l, are *both* circular and neutral, we derive the following inequality:

$$g_l^{(2)} \geqslant (1/p) \sum_{j=0}^{l-1} \gamma^{l-j} g_l^{(2)} + \gamma^l g_{D_l}^{(2)} \tag{4}$$

where

$$g_{D_l}^{(2)} \equiv [\ln Z_{D_l}^{(2)}(D_l, M_l = \rho|D_l|; \beta)]/|D_l|$$

However, from Jensen's inequality, we have

$$\begin{aligned}\ln Z_{D_l}^{(2)} &= \ln(|D_l|^{M_l}/M_l!) + \left\{\int\cdots\int_{D_l^{M_l}} d^2x_1 \cdots d^2x_{M_l}\right.\\ &\quad \left.\times \exp[-\beta H^{(2)}(x_1,\ldots,x_{M_l})]\right\}\Big/\int\cdots\int_{D_l^{M_l}} d^2x_1\cdots d^2x_{M_l}\\ &\geqslant \ln(|D_l|^{M_l}/M_l!) - (\beta/|D_l|^{M_l})\int\cdots\int_{D_l^{M_l}} d^2x_1\cdots d^2x_{M_l} H^{(2)}(x_1,\ldots,x_{M_l})\\ &= \ln(|D_l|^{M_l}/M_l!) + (\beta/M_l) H_{bb,D_l}^{(2)}\end{aligned}$$

where $H_{bb,D_l}^{(2)}$ is the background self-energy in D_l. At this stage the peculiarity of the two-dimensional case appears: In the three-dimensional case $H_{bb}^{(3)}$ is positive. However, let us denote by $\tilde{R}_l$ the smallest radius of the disk that contains D_l. Then a direct computation shows that

$$\beta H_{bb,D_l}^{(2)}/M_l \geqslant -\tfrac{1}{2}\beta\rho e^2|D_l| \ln \tilde{R}_l$$

Taking $\tilde{R}_l = R_l$, and for large l, the inequality (4) implies that

$$g_l^{(2)} \geqslant \frac{1}{p}\sum_{j=0}^{l-1} \gamma^{l-j} g_j^{(2)} + \gamma^l \rho\left(1 - \ln\rho - \frac{\beta e^2}{4}\ln\frac{(1+p)^{2l}}{\pi\rho}\right)$$

Writing $\gamma^l\rho\{1 - \ln\rho - \frac{1}{4}\beta e^2 \ln[(1+p)^{2l}/\pi\rho]\}$ simply as $C_l^{(2)}$, $(\sum_{l=0}^{\infty} C_l^{(2)} < \infty)$, and putting

$$g_l^{(2)} = \frac{1}{p}\sum_{j=0}^{l-1} \gamma^{l-j} g_j^{(2)} + C_l^{(2)} + \delta_l^{(2)}$$

where $\delta_l^{(2)}$ is a nonnegative real number, we can prove the explicit solution of this equation, valid for $l > 0$, by induction to be

$$g_l^{(2)} = (1-\gamma) g_0^{(2)} + \gamma(C_l^{(2)} + \delta_l^{(2)}) + (1-\gamma)\sum_{j=1}^{l}(C_j^{(2)} + \delta_j^{(2)})$$

$g_l^{(2)}$ has a limit $g^{(2)}(\rho, \beta)$ because (i) $g_0^{(2)}$ is finite; (ii) the terms involving $C^{(2)}$ have a limit; (iii) since each $\delta_j^{(2)} \geqslant 0$ and since $g_l^{(2)}$ has an upper bound by H-stability, the sum involving the $\delta^{(2)}$ must converge and then $\delta_l^{(2)} \to 0$.

This limit has been established evidently for a ρ-dependent sequence of disks. However, one can show that the thermodynamic limit of $g_l^{(2)}$ exists for more general sequences of domains $\{\Lambda_l\}$ and is the same as that for the standard sequence (disks). A discussion of conditions on domains $\{\Lambda_l\}$ may be found in Ref. 4.

Remark. Recently, Fröhlich[6] has proved via constructive quantum field theory the existence of the thermodynamic limit of the grand canonical free energy density for the two-component plasma in two dimensions, for $\beta < 4/e^2$. It should be interesting to obtain this result directly by the methods of the statistical mechanics for the canonical ensemble (for this model, the equivalence of ensembles introduces no problem). Letting $Z_{(2)}^{(2)}$ be the configuration partition function (cpf) of the two-component plasma of N particles of charge $(-e)$ and coordinates (x_i) and N particles of charge $(+e)$ and coordinates (y_j), $Z_{(1)}^{(2)}$ the cpf of the one-component plasma with N particles of charge $(-e)$ in a neutralizing background, and $Z_0^{(2)}$ the cpf of a gas of N particles without interaction, all of volume Λ, we have immediately, using Jensen's inequality $[(\tilde{x}_N) \equiv (x_1,..., x_N); (\tilde{y}_N) \equiv (y_1,..., y_N)]$,

$$Z_{(2)}^{(2)} = \frac{|\Lambda|^N}{(N!)^2} \int \cdots \int_{\Lambda^N} d^2\tilde{x}_N \exp[-\beta H^-(\tilde{x}_N)]$$

$$\times \frac{\int \cdots \int_{\Lambda^N} d^2\tilde{y}_N \exp[-\beta H^+(\tilde{y}_N) - \beta H^{+-}(\tilde{x}_N; \tilde{y}_N)]}{\int \cdots \int_{\Lambda^N} d^2\tilde{y}_N}$$

$$\geqslant \frac{|\Lambda|^N}{(N!)^2} \int \cdots \int_{\Lambda^N} d^2\tilde{x}_N \exp\{-\beta[H^-(\tilde{x}_N) + \langle H^+(\tilde{y}_N) + H^{+-}(\tilde{x}_N; \tilde{y}_N)\rangle]\}$$

where

$$\langle H^+(\tilde{y}_N)\rangle = -\frac{e^2}{2} \sum_{i \neq j}^{N} \frac{\int \cdots \int_{\Lambda^N} d^2y_1 \cdots d^2y_N \ln|y_i - y_j|}{\int \cdots \int_{\Lambda^N} d^2y_1 \cdots d^2y_N}$$

$$= -\frac{e^2\rho^2}{2}\left(1 - \frac{1}{N}\right) \iint_{\Lambda^2} d^2y_1\, d^2y_2 \ln|y_1 - y_2|$$

and

$$\langle H^{+-}(\tilde{x}_N; \tilde{y}_N)\rangle = e^2 \sum_{i,j=1}^{N} \frac{\int \cdots \int_{\Lambda^N} d^2y_1 \cdots d^2y_N \ln|x_i - y_j|}{\int \cdots \int_{\Lambda^N} d^2y_1 \cdots d^2y_N}$$

$$= \rho e^2 \sum_{i=1}^{N} \int_{\Lambda} d^2y \ln|x_i - y|$$

so that for large Λ

$$Z_{(2)}^{(2)} \geqslant Z_{(0)}^{(2)} Z_{(1)}^{(2)}$$

The difficulty is then to find an extensive upper bound on $\ln Z_{(2)}^{(2)}$. It is expected that $\ln Z_{(2)}^{(2)}$ is bounded above, for $T > T_c = e^2/4k_B$, by an extensive quantity plus a contribution proportional to $N \ln Q^*$, where Q^* is the partition function of the system in a volume unit with two particles $(+e)$ and $(-e)$. If this conjecture is verified, the proof of the existence of the thermodynamic limit should be obtained immediately by the application of the strategy of Lieb and Lebowitz.[4]

5. SCALING PROPERTIES

As first observed by Brush *et al.*,[7] the thermodynamic quantities of the three-dimensional, one-component classical plasma depend essentially only on the plasma parameter $\Gamma \equiv \beta e^2(4\pi\rho/3)^{1/3}$. Similarly, for the two-dimensional case, a scaling property can be established, leading immediately to the equation of state derived by Hauge and Hemmer,[8] independently of the knowledge of the existence of the thermodynamic limit for the canonical free energy.

More generally, we will now show that the state of the system can be characterized essentially by the plasma parameter $\gamma^{(\nu)} \equiv \beta e^2 \rho^{(\nu-2)/\nu}$.

Let N, the number of particles, be kept fixed; and consider the following transformations:

$$x' = xh^{(\nu)}, \qquad \beta' = \beta m^{(\nu)}, \qquad \rho' = \rho n^{(\nu)}$$

Then $h^{(\nu)} = (n^{(\nu)})^{-1/\nu}$ and

$$\int_{\Lambda^N}\!\!\cdots\!\int \prod_{i=1}^{N} d^\nu x_i \exp[-\beta H^{(\nu)}(x)]$$
$$= \int_{\Lambda'^N}\!\!\cdots\!\int \prod_{i=1}^{N} d^\nu x_i' \left\{\exp - \frac{\beta' H^{(\nu)}(x')}{m^{(\nu)}(n^{(\nu)})^{(\nu-2)/\nu}}\right\}(n^{(\nu)})^N\{\exp[-\beta' N(\ln n^{(\nu)})\delta_{\nu,2}]\}$$

The choice $m^{(\nu)}(n^{(\nu)})^{(\nu-2)/2} = 1$ gives therefore for the free energy density times $(-\beta)$

$$g^{(\nu)}(\rho, \beta) = \frac{1}{n^{(\nu)}}\, g^{(\nu)}(\rho n^{(\nu)}, \beta(n^{(\nu)})^{(2-\nu)/\nu}) + (\rho \ln n^{(\nu)})\left[1 - \frac{\beta e^2}{4}(n^{(\nu)})^{(2-\nu)/\nu}\,\delta_{\nu,2}\right]$$

Choosing $\rho n^{(\nu)} = 1$, we have

$$g^{(\nu)}(\rho, \beta) = \rho g^{(\nu)}(1, \gamma^{(\nu)}) - (\rho \ln \rho)(1 - \tfrac{1}{4}\gamma^{(\nu)}\,\delta_{\nu,2}) \tag{5}$$

and the equation of state reads

$$\beta P = -\rho^2 \frac{\partial}{\partial \rho}\left(\frac{g^{(\nu)}(\rho, \beta)}{\rho}\right) = \beta\rho\left(\frac{\nu-2}{\nu}\right)\langle h^{(\nu)}\rangle + \rho\left(1 - \frac{\gamma^{(\nu)}}{4}\,\delta_{\nu,2}\right) \tag{6}$$

where $\langle h^{(\nu)}\rangle$ is the mean value of the potential energy per particle. For $\nu = 1$, taking account of the bound of H-stability, we obtain $\beta P/\rho \leqslant 1 - \beta e^2/12\rho$; and for $\nu = 2$, $\beta P/\rho = 1 - \beta e^2/4$. However, we now obtain, in this dimensionality, one more result. In fact, since the existence of the thermodynamic limit of the free energy density has been established, this takes the simple form

$$-\beta f^{(2)}(\beta e^2) = -\beta f^{*(2)}(\beta e^2) - (\rho \ln \rho)(1 - \tfrac{1}{4}\beta e^2)$$

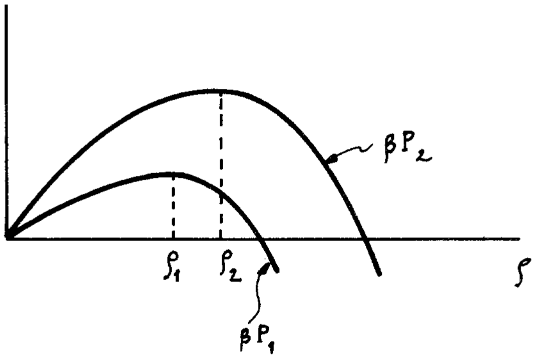

Fig. 1

where $f^{*(2)}(\frac{1}{4}\beta e^2 = 1)$ is the free energy density at the "catastrophe point" or "threshold of thermodynamic instability" $T_c = e^2/4k_B$. The pressure and compressibility become negative for $T < T_c$. On the other hand, for $\nu = 3$, $\beta P/\rho \geqslant 1 - 3\Gamma/10$ (from the H-stability). Thus, for $\Gamma < 10/3$ the pressure is positive. Moreover, using the results of Theorem 2, we know that the excess free energy density behaves as $-\rho^{4/3}$, and thus the pressure must be negative for large Γ. Assuming that the free energy density can be approximated by either of the bounds previously given, we define the corresponding pressures (Fig. 1):

$$\beta P_1 = \rho(1 - \tfrac{3}{10}\Gamma), \qquad \beta P_2 = \rho(1 - \tfrac{2}{9}\Gamma)$$

The values of $\Gamma_1 = 2.5$ and $\Gamma_2 = 3.375$, which correspond to the maxima of the pressures P_1 and P_2, respectively, are not far from the value $\Gamma \sim 3$ from Monte Carlo calculations, which defines the onset of a negative compressibility.

To conclude, we note that it is widely accepted that an ordering, namely a transition into a crystalline state, is possible in this system for $\nu \geqslant 2$. Then it should be worthwhile to prove or disprove this in the region of $\gamma^{(\nu)}$ where the model satisfies thermodynamic stability.

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